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The Minimal Normal Extension Problem for Subnormal Operators

JAMES DUDZIAK

Department of Mathematics, Bucknell University, Lewisburg Pennsylvania 17837

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INTRODUCTION

The first section of this paper sets forth a functional calculus for subnormal operators and then poses a rather natural operator-theoretic problem about this calculus. The second section shows that the operator-theoretic problem is equivalent to a function-theoretic problem concerning the weak* density of a certain algebra of functions in a certain L^∞ space. The third section proves that this algebra of functions is weak* dense over an open subset of the complex plane, thus shifting interest to that part of the algebra not living over this subset. The fourth section solves the weak* density problem totally in the hypodirichlet case. Finally, the fifth section poses some open questions and presents some relevant examples concerning the weak* density problem in general.

We adopt the following conventions and notation.

All Hilbert spaces are complex and separable and all operators live on Hilbert space and are bounded and linear. For $\mathscr{K}$ a Hilbert space $\mathscr{B}(\mathscr{K})$ denotes the space of all operators on $\mathscr{K}$. Given $A \in \mathscr{B}(\mathscr{K})$, $\sigma(A)$ denotes the spectrum of A and for $\mathscr{H}$ a closed subspace of $\mathscr{K}$, $A|\mathscr{H}$ denotes the restriction of A to $\mathscr{H}$. WOT stands for the weak operator topology.

For C a subset of the complex plane $\mathbb{C}$, let C^0, $\bar{C}$, and ∂C denote the interior, closure, and boundary of C, respectively. The characteristic function of C is denoted X_C and the restriction of a function f to C is denoted $f|C$. Given $z_0 \in \mathbb{C}$ and $\delta > 0$, $\Delta(z_0; \delta)$ denotes the open disc of radius δ centered at z_0.

All measures are compactly supported regular complex Borel measures. Denote the support of a measure μ by $\operatorname{spt}\mu$ and the restriction of μ to a Borel set C by $\mu|C$. When μ is absolutely continuous with respect to another measure ν we write $\mu \ll \nu$ and when μ and ν are mutually absolutely

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continuous we write $\mu \approx \nu$. We think of $L^1(\mu)$ as the space of all measures ν such that $\nu \ll \mu$.

Finally, for K a compact subset of $\mathbb{C}$, $C(K)$ denotes the space of all continuous complex-valued functions on K while $R(K)$ denotes the uniform closure in $C(K)$ of the rational functions in z with poles off K.

I. Statement of the Problem

An operator S on a Hilbert space $\mathscr{H}$ is called *subnormal* iff there exists a Hilbert space $\mathscr{K}$ containing $\mathscr{H}$ and a normal operator N on $\mathscr{K}$ such that $N\mathscr{H} \subseteq \mathscr{H}$ and $N|\mathscr{H} = S$. Such an N, called a *normal extension of* S, is called a *minimal normal extension* (*mne*) *of* S iff the smallest closed subspace of $\mathscr{K}$ containing $\mathscr{H}$ and reducing N is $\mathscr{K}$ itself. A useful fact is the following.

I.1. Proposition. *For S subnormal on $\mathscr{H}$ with normal extension N on $\mathscr{K}$, N is a* mne *of S iff $\mathscr{K} =$ closed span $\{N^{*n}x\colon n \geqslant 0, x \in \mathscr{H}\}$* [C, III.2.4].

Any two mnes of S are unitarily equivalent in a manner that fixes S [C, III.2.5], so any mne of S is referred to as *the* mne of S. There is the following relationship between spectra.

I.2. Proposition. *For S subnormal and $N =$* mne *S, $\sigma(S)$ is the union of $\sigma(N)$ and some collection of bounded components of $\mathbb{C}\backslash\sigma(N)$* [C, III.2.11].

The functional calculus for a subnormal operator considered in this paper is constructed from the functional calculus for the mne of the subnormal. The properties of the latter calculus are stated below but first some definitions and facts are needed.

Given two spaces equipped with weak* topologies, a map between them which is a weak*–weak* homeomorphism shall be more briefly referred to as a *weak* homeomorphism*. Recall that L^∞, being the dual of L^1, has a weak* topology and that the space of operators, being the dual of the space of trace-class operators, has a weak* topology also. Finally, for N a normal operator on $\mathscr{K}$, let $W^*(N)$ denote the von Neumann algebra generated by N, i.e., let $W^*(N)$ denote the weak* closure in $\mathscr{B}(\mathscr{K})$ of the polynomials in N and N^*.

I.3. Theorem. *If N is a normal operator on a Hilbert space $\mathscr{K}$, then there exists a measure μ with support $\sigma(N)$ and a map $\Phi\colon L^\infty(\mu) \to W^*(N)$ that is an isometric $*$-isomorphism and a weak* homeomorphism sending the identity function z to N. Furthermore, if ν is a measure on $\mathbb{C}$ and $\Psi\colon L^\infty(\nu) \to \mathscr{B}(\mathscr{K})$ is a weak*-*WOT *continuous, one-to-one, $*$-homomorphism* *sending z to N, then $\nu \approx \mu$, so $L^\infty(\nu) = L^\infty(\mu)$ in all respects, and $\Psi = \Phi$* [C, II.7.6].

Any measure μ as in I.3 is called a *scalar-valued spectral measure* (*svsm*) *for N* and the operator $\Phi(f)$ in I.3 is denoted $f(N)$. Note that svsms are unique up to mutual absolute continuity. A svsm for the mne of a subnormal S shall be more briefly referred to as a *svsm for S*.

Our functional calculus for subnormals arises in the following way. Given S subnormal on $\mathscr{H}$ with mne N and svsm μ, by I.2 and I.3 spt $\mu = \sigma(N) \subseteq \sigma(S)$. Thus the set of rational functions in z with poles off $\sigma(S)$ is contained in $L^\infty(\mu)$. Denote the weak* closure in $L^\infty(\mu)$ of this set by $R^\infty(\sigma(S), \mu)$. Denote the weak* closure in $\mathscr{B}(\mathscr{H})$ of the rational functions in S with poles off $\sigma(S)$ by $R^\infty(S)$. For $z_0 \notin \sigma(S)$, one may easily verify that $(N-z_0)^{-1}\, \mathscr{H} \subseteq \mathscr{H}$ and $(N-z_0)^{-1} | \mathscr{H} = (S-z_0)^{-1} \in R^\infty(S)$. Algebra and a limiting argument now show that $f(N)\, \mathscr{H} \subseteq \mathscr{H}$ and $f(N)|\mathscr{H} \in R^\infty(S)$ for each $f \in R^\infty(\sigma(S), \mu)$. Setting $f(S) \equiv f(N)|\mathscr{H}$, we obtain our functional calculus for S. Its properties are noted in the following.

I.4. THEOREM. *If S is a subnormal operator on a Hilbert space $\mathscr{H}$ and μ is a svsm for S, then the map $f \in R^\infty(\sigma(S), \mu) \to f(S) \in R^\infty(S)$ is an isometric isomorphism and a weak* homeomorphism sending the identity function z to S. Furthermore, if Φ: $R^\infty(\sigma(S), \mu) \to \mathscr{B}(\mathscr{H})$ is a weak*-WOT continuous homomorphism sending z to S, then $\Phi(f) = f(S)$ for each $f \in R^\infty(\sigma(S), \mu)$* [C, III.12.10].

Clearly $f(\text{mne } S)$ is a normal extension of $f(S)$ for any $f \in R^\infty(\sigma(S), \mu)$. This paper addresses the problem of determining when $f(\text{mne } S) = \text{mne } f(S)$. Note that for $f(z) \equiv z$, $f(\text{mne } S) = \text{mne } f(S)$ always, while for $f(z) \equiv 1$, $f(\text{mne } S) \neq \text{mne } f(S)$ whenever S is nonnormal. Vaguely speaking it seems to be the case that $f(\text{mne } S) = \text{mne } f(S)$ precisely when f has small level sets in some suitable sense.

Robert Olin investigated this problem for functions in $R(\sigma(S))$ (see [O] or [C, VIII.2]). Later, he and John Conway investigated this problem for functions in $P^\infty(\mu) \equiv$ the weak* closure in $L^\infty(\mu)$ of the polynomials in z (see [CO] or [C, VIII. 2]). Indeed, this paper was directly inspired by [CO], many of its results and techniques being adaptations of those in [CO]. Finally, the author would like to express his thanks to Professor James Thomson of Virginia Polytechnic Institute for a splitting argument used in the proofs of IV.9 and V.1 below.

II. A FUNCTION-THEORETIC EQUIVALENT

Before presenting our function-theoretic equivalent a few preliminaries are necessary.

For K a compact subset of $\mathbb{C}$ and μ a measure on K, let $R^2(K, \mu)$ denote the closure in $L^2(\mu)$ of $R(K)$, or, what is the same thing, the closure in $L^2(\mu)$ of the rational functions in z with poles off K. Clearly $zR^2(K, \mu) \subseteq R^2(K, \mu)$, so one may define the operator $S_{K,\mu}$ of multiplication by z on $R^2(K, \mu)$. The operator N_μ of multiplication by z on $L^2(\mu)$ is a normal extension of $S_{K,\mu}$. The Stone–Weierstrass theorem and I.1 imply that $N_\mu = \text{mne}\, S_{K,\mu}$. Via I.3 one sees that μ is a svsm for N_μ and that $f(N_\mu)$ is the operator of multiplication by f on $L^2(\mu)$. Thus for $f \in R^\infty(\sigma(S_{K,\mu}), \mu)$, $f(S_{K,\mu})$ is the operator of multiplication by f on $R^2(K, \mu)$. Let $R^\infty(K, \mu)$ denote the weak* closure in $L^\infty(\mu)$ of $R(K)$, or, what is the same thing, the weak* closure in $L^\infty(\mu)$ of the rational functions in z with poles off K. Clearly $\sigma(S_{K,\mu}) \subseteq K$, so $R^\infty(K, \mu) \subseteq R^\infty(\sigma(S_{K,\mu}), \mu)$. Thus $f(S_{K,\mu})$ is well-defined for each $f \in R^\infty(K, \mu)$.

Recall that an operator A on a Hilbert space $\mathscr{H}$ is *rationally cyclic* iff there is a vector $x_0 \in \mathscr{H}$ such that $\mathscr{H}$ is the closure of $\{f(A)\, x_0 : f$ is a rational function in z with poles off $\sigma(A)\}$. Clearly $S_{K,\mu}$ is a rationally cyclic subnormal. Any rationally cyclic subnormal is unitarily equivalent to some $S_{K,\mu}$ [C, III.5.2].

Let $\mathscr{A}$ be a linear manifold in $L^\infty(\mu)$. Clearly $\bar{\mathscr{A}}^{\text{wk}^* L^\infty(\mu)} \subseteq \bar{\mathscr{A}}^{L^2(\mu)} \cap L^\infty(\mu)$. The containment can be proper. Note, however, that for $\nu \approx \mu$, $\bar{\mathscr{A}}^{\text{wk}^* L^\infty(\nu)} = \bar{\mathscr{A}}^{\text{wk}^* L^\infty(\mu)}$. Thus $\bar{\mathscr{A}}^{\text{wk}^* L^\infty(\mu)} \subseteq \bigcap_{\nu \approx \mu} \bar{\mathscr{A}}^{L^2(\nu)} \cap L^\infty(\nu)$. This containment can never be proper.

II.1. PROPOSITION. $\bar{\mathscr{A}}^{\text{wk}^* L^\infty(\mu)} = \bigcap_{\nu \approx \mu} \bar{\mathscr{A}}^{L^2(\nu)} \cap L^\infty(\nu)$ [BOT, 2.2].

We may now present our function-theoretic equivalent.

II.2. THEOREM. *Suppose K is a compact subset of $\mathbb{C}$, μ is a measure on K, and $f \in R^\infty(K, \mu)$. Set $\mathscr{S}(K, \mu) \equiv$ the set of subnormal operators S with $\sigma(S) \subseteq K$ and* svsm *μ. Then the following are equivalent*:

(a) *the algebra generated by $R^\infty(K, \mu)$ and $\bar{f}$ is weak* dense in $L^\infty(\mu)$*,

(b) $f(\text{mne}\ S) = \text{mne}\, f(S)$ *for each $S \in \mathscr{S}(K, \mu)$, and*

(c) $f(\text{mne}\ S) = \text{mne}\, f(S)$ *for each rationally cyclic $S \in \mathscr{S}(K, \mu)$.*

Proof. Denote the algebra generated by $R^\infty(K, \mu)$ and $\bar{f}$ by $\mathscr{A}$.

(a) $\Rightarrow$ (b): Let $S \in \mathscr{S}(K, \mu)$ on $\mathscr{H}$ have mne N on $\mathscr{K}$. Suppose $y \in \mathscr{K}$ is orthogonal to $\{f(N)^{*n} x \colon n \geqslant 0, x \in \mathscr{H}\}$.

Fix $x \in \mathscr{H}$. Since $S \in \mathscr{S}(K, \mu)$, $R^\infty(K, \mu) \subseteq R^\infty(\sigma(S), \mu)$ and so $g(N)\, x = g(S) x \in \mathscr{H}$ whenever $g \in R^\infty(K, \mu)$. Clearly then $\langle (\bar{f}^n g)(N)\, x, y \rangle = \langle f(N)^{*n} g(N) x, y \rangle = 0$ for $n \geqslant 0$ and $g \in R^\infty(K, \mu)$. Thus the linear functional $h \in L^\infty(\mu) \to \langle h(N)\, x, y \rangle \in \mathbb{C}$ vanishes for $h \in \mathscr{A}$. By I.3 this functional is weak* continuous. Now (a) simply asserts that $\mathscr{A}$ is weak* dense in $L^\infty(\mu)$. Hence the functional vanishes identically on $L^\infty(\mu)$. In particular, $\langle N^{*n}x, y\rangle = \langle(\bar{z}^n)(N)x, y\rangle = 0$ for $n \geqslant 0$.

We conclude that $y \in \mathscr{H}$ is orthogonal to the closed span of $\{N^{*n}x: n \geqslant 0, x \in \mathscr{H}\}$. As $N = \text{mne } S$, I.1 implies that $y = 0$. By the Hahn–Banach theorem, $\mathscr{H} = \text{closed span } \{f(N)^{*n}x: n \geqslant 0, x \in \mathscr{H}\}$. Using I.1 once more, $f(N) = \text{mne } f(S)$, i.e., (b).

(b) $\Rightarrow$ (c): Trivial.

(c) $\Rightarrow$ (a): Let $\nu \approx \mu$. Clearly $S_{K,\nu} \in \mathscr{S}(K, \mu)$. Thus by (c), $f(N_\nu) = \text{mne } f(S_{K,\nu})$. From I.1 we conclude that $L^2(\nu) = \text{closed span } \{f(N_\nu)^{*n}g: n \geqslant 0, g \in R^2(K, \nu)\} = \text{closed span } \{\bar{f}^n g: n \geqslant 0, g \in R^2(K, \nu)\}$. Since the closure in $L^2(\nu)$ of $R^\infty(K, \nu) = R^\infty(K, \mu)$ is $R^2(K, \nu)$, it follows that $L^2(\nu) = \text{closed span } \{\bar{f}^n g: n \geqslant 0, g \in R^\infty(K, \mu)\} = \bar{\mathscr{A}}^{L^2(\nu)}$. Hence $L^\infty(\mu) = L^\infty(\nu) = L^2(\nu) \cap L^\infty(\nu) = \bar{\mathscr{A}}^{L^2(\nu)} \cap L^\infty(\nu)$. As $\nu \approx \mu$ was otherwise arbitrary, $L^\infty(\mu) = \bigcap_{\nu \approx \mu} \bar{\mathscr{A}}^{L^2(\nu)} \cap L^\infty(\nu)$. By II.1, $\bar{\mathscr{A}}^{\text{wk}^* L^\infty(\mu)} = L^\infty(\mu)$, i.e., (a). ▌

The corresponding equivalence of (a) and (b) in Conway and Olin's $P^\infty(\mu)$ paper [CO, 5.3] uses in its proof von Neumann's double commutant theorem and a theorem of Sarason concerning algebras of normal operators. The simpler and more elegant result of Ball, Olin, and Thomson, II.1 above, takes the place of these two theorems in our proof and also yields the equivalence of (c) with (a) and (b).

For the rest of this paper our concern will no longer be with the minimal normal extension problem but with the weak* density problem in II.2 equivalent to it.

III. L^∞ Splitting over Domains of Analyticity

For now and forevermore, K is a compact subset of $\mathbb{C}$, μ is a measure on K, and $f \in R^\infty(K, \mu)$. Set $\mathscr{A} \equiv$ the algebra generated by $R^\infty(K, \mu)$ and $\bar{f}$. It will be convenient, given $\nu \ll \mu$, to set $\mathscr{A}^\infty(\nu) \equiv$ the weak* closure in $L^\infty(\nu)$ of $\mathscr{A}$ (this makes sense since $\nu \ll \mu$ implies that $L^\infty(\mu) \subseteq L^\infty(\nu)$). We seek to determine when $\mathscr{A}^\infty(\mu) = L^\infty(\mu)$.

Before presenting our result concerning L^∞ splitting over domains of analyticity a few preliminaries are necessary. These have to do with some work of Chaumat and a theorem of Bishop. Recall that we say a measure ν on X *annihilates* $A \subseteq C(X)$ and write $\nu \perp A$ or $\nu \in A^\perp$ iff $\int g\,d\nu = 0$ for each $g \in A$.

III.1. Theorem. *There exists a Borel set $\Delta_\perp$ and a measure $\mu_\perp \in R(K)^\perp$ such that $R^\infty(K, \mu) = R(K, \mu | \Delta_\perp) \oplus L^\infty(\mu | \mathbb{C} \backslash \Delta_\perp)$ and $\mu | \Delta_\perp \approx \mu_\perp$* [Ch, I.2 *and its proof*].

If $\mu_\perp = 0$, then $R^\infty(K, \mu) = L^\infty(\mu)$. Thus by II.2, $f(\text{mne } S) = \text{mne } f(S)$ for each $S \in \mathscr{S}(K, \mu)$ with no hypothesis on f necessary besides being in $R^\infty(K, \mu)$. This phenomenon arises for a trivial reason, however: each $S \in \mathscr{S}(K, \mu)$, and so, too, $f(S)$, is normal (as can be seen by contemplating $f(z) \equiv 1$).

Consequently, for now and forevermore, let $\Delta_\perp$ and $\mu_\perp$ be as in III.1 and assume $\mu_\perp \neq 0$. Clearly $\mathscr{A}^\infty(\mu) = \mathscr{A}^\infty(\mu | \Delta_\perp) \oplus L^\infty(\mu | \mathbb{C} \backslash \Delta_\perp)$ and $\mathscr{A}^\infty(\mu | \Delta_\perp) = \mathscr{A}^\infty(\mu_\perp)$. Thus we now seek to determine when $\mathscr{A}^\infty(\mu_\perp) = L^\infty(\mu_\perp)$. Note that $R^\infty(K, \mu_\perp)$ has no nontrivial L^∞ summand, i.e., $|\mu_\perp|(\mathbb{C} \backslash \Delta) = 0$ whenever $R^\infty(K, \mu_\perp) = R^\infty(K, \mu_\perp | \Delta) \oplus L(\mu_\perp | \mathbb{C} \backslash \Delta)$.

The *envelope* E of μ with respect to K is the set of points $z \in K$ such that there exists a measure $\mu_z \ll \mu$ such that $\mu_z(\{z\}) = 0$ and $\int g\, d\mu_z = g(z)$ for each $g \in R(K)$. Setting $\nu(\zeta) \equiv (\zeta - z)\, \mu_z(\zeta)$, we have $\nu \perp R(K)$ and $\nu \ll \mu$. Thus $\int g\, d\nu = 0$ for each $g \in R^\infty(K, \mu)$. As $L^\infty(\mu | \mathbb{C} \backslash \Delta_\perp)$ is a direct summand of $R^\infty(K, \mu)$, $\nu \ll \mu | \Delta_\perp$ and so $\mu_z \ll \mu_\perp$. Consequently the number $\check{g}(z) \equiv \int g\, d\mu_z$ is defined for $g \in R^\infty(K, \mu_\perp)$. Clearly $\check{g}(z)$ is independent of the particular μ_z chosen and so well-defined. We thus have a homomorphism $g \to \check{g}$, called *Chaumat's map* for μ and K, which associates to each function in $R^\infty(K, \mu_\perp)$ a point function on E. Five facts about the envelope and Chaumat's map are needed.

III.2. Proposition. (a) *E is a Borel set* [Ch, VI.6].

(b) $E^0 = (\bar{E})^0$ [Ch, VI.1].

(c) *$\bar{E}$ is the union of* $\operatorname{spt}\mu_\perp$ *and some collection of bounded components of* $\mathbb{C} \backslash \operatorname{spt}\mu_\perp$ [Ch, VI.10].

(d) *For each $g \in R^\infty(K, \mu_\perp)$, $\check{g}$ is analytic on E^0* [Ch, III.1, III.3, IV.3, *and Montel's theorem*].

(e) *Suppose $g \in R^\infty(K, \mu_\perp)$ and $z_0 \in \mathbb{C}$ are such that $\check{g}$ has an analytic extension $\tilde{g}$ to a neighborhood of z_0. Then there is a function $h \in R^\infty(K, \mu_\perp)$ such that $g(z) - \tilde{g}(z_0) = h(z)(z - z_0)$* [Ch, IV.2 *and the proof of* VII.4].

Let λ_E denote area measure on E. The author cannot resist mentioning in passing that Chaumat has shown the map $\check{}$ to be an isometric isomorphism and a weak* homeomorphism of $R^\infty(K, \mu_\perp)$ onto $R^\infty(K, \lambda_E)$ which is the map $g \to g | E$ on $R(K)$ [Ch, IV.2].

Let X be a compact Hausdorff space and let A be a uniformly closed subalgebra of $C(X)$ containing the constants. A subset $\mathscr{M}$ of X is called an *antisymmetric set* for A iff every function in A which is real-valued on $\mathscr{M}$ is constant on $\mathscr{M}$. Maximal antisymmetric sets for A are easily seen to exist and form a closed partition of X. Less easily seen is the following theorem of Bishop which is a generalization of the Stone–Weierstrass theorem.

III.3. THEOREM. *If $F \in C(X)$ is such that $F|\mathscr{M} \in A|\mathscr{M}$ whenever $\mathscr{M}$ is a maximal antisymmetric set for A, then $F \in A$* [S, 12.1].

We may now present our result concerning L^∞ splitting over domains of analyticity.

III.4. THEOREM. *Set $U \equiv \{z_0 \in \mathbb{C}\colon \tilde{f}$ has a nonconstant analytic extension to a neighborhood of $z_0\}$. Then $\mathscr{A}^\infty(\mu_\perp) = \mathscr{A}^\infty(\mu_\perp|\mathbb{C}\backslash U) \oplus L^\infty(\mu_\perp\,|\,U)$.*

Proof. Set $X \equiv$ the maximal ideal space of $L^\infty(\mu_\perp)$, $\hat{\ }\equiv$ the Gelfand transform of $L^\infty(\mu_\perp)$ onto $C(X)$, and $A \equiv \widehat{\mathscr{A}^\infty(\mu_\perp)}$. Suppose $\mathscr{M}$ is a maximal antisymmetric set for A such that $\hat{z}(\mathscr{M}) \cap U \neq \varnothing$. Then there exist $\phi \in \mathscr{M}$, $z_0 \in \mathbb{C}$ such that $\phi(z) = z_0$, and a nonconstant analytic extension $\tilde{f}$ of f to a neighborhood of z_0. Write $\tilde{f}(z) - \tilde{f}(z_0)$ as $\alpha\tilde{g}(z)(z-z_0)^n$, where $|\alpha| = 1$, $\tilde{g}(z_0) > 0$, and $n \geqslant 1$. Applying III.2(e) repeatedly, there exists a $g \in R^\infty(K, \mu_\perp)$ such that

$$f(z) - \tilde{f}(z_0) = \alpha g(z)(z-z_0)^n. \tag{$*$}$$

Clearly $\tilde{g}$ is an analytic extension of g to a neighborhood of z_0. Applying III.2(c) again, there exists an $h \in R^\infty(K, \mu_\perp)$ such that

$$g(z) - \tilde{g}(z_0) = h(z)(z-z_0). \tag{$**$}$$

Since $\hat{f}$ and $\overline{\hat{f}} \in A$ and $\mathscr{M}$ is a maximal antisymmetric set for A, $\hat{f}$ is constant on $\mathscr{M}$. As $\phi \in \mathscr{M}$, $\hat{f} \equiv \phi(f)$ on $\mathscr{M}$. As $\phi(z) = z_0$, $(*)$ implies that $\phi(f) = \tilde{f}(z_0)$. Thus $\hat{f} \equiv \tilde{f}(z_0)$ on $\mathscr{M}$. But then $(*)$ implies that

$$\psi(g) = 0 \text{ or } \psi(z) = z_0 \text{ whenever } \psi \in \mathscr{M}. \tag{$***$}$$

Because of $(**)$, $\psi(g) = \tilde{g}(z_0)$ whenever $\psi(z) = z_0$. But then, because of $(***)$, $\psi(g) = 0$ or $\tilde{g}(z_0)$ whenever $\psi \in \mathscr{M}$, i.e., $\hat{g}$ is real-valued on $\mathscr{M}$. Since $\hat{g} \in A$ and $\mathscr{M}$ is a maximal antisymmetric set for A, $\hat{g}$ is constant on $\mathscr{M}$. As $\phi \in \mathscr{M}$, $\hat{g} \equiv \phi(g)$ on $\mathscr{M}$. As $\phi(z) = z_0$, $(**)$ implies that $\phi(g) = \tilde{g}(z_0)$. Thus $\hat{g} \equiv \tilde{g}(z_0) \neq 0$ on $\mathscr{M}$. But then $(***)$ implies that $\psi(z) = z_0$ whenever $\psi \in \mathscr{M}$, i.e., $\hat{z}(\mathscr{M}) = \{z_0\}$.

The proof so far has thus shown that $\hat{z}(\mathscr{M})$ is a singleton whenever $\mathscr{M}$ is a maximal antisymmetric set for A such that $\hat{z}(\mathscr{M}) \cap U \neq \varnothing$. Set $C_0(U) \equiv$ the space of continuous functions on $\mathbb{C}$ vanishing off U. For $g \in C_0(U)$ and $\phi \in X$, $\hat{g}(\phi) = g(\phi(z))$. It now follows easily that any function in $\widehat{C_0(U)}$ is constant on the maximal antisymmetric sets for A. By III.3, $\widehat{C_0(U)} \subseteq A = \widehat{\mathscr{A}^\infty(\mu_\perp)}$. Thus $C_0(U) \subseteq \mathscr{A}^\infty(\mu_\perp)$. Let ν be a weak* continuous annihilator of $\mathscr{A}^\infty(\mu_\perp)$. Then ν annihilates $C_0(U)$. As U is clearly open, it follows that $|\nu|(U) = 0$. Thus ν is a weak* continuous annihilator of $\mathscr{A}^\infty(\mu_\perp|\mathbb{C}\backslash U) \oplus L^\infty(\mu_\perp\,|\,U)$. By the Hahn–Banach theorem,

$\mathscr{A}^\infty(\mu_\perp \mid \mathbb{C}\backslash U) \oplus L^\infty(\mu_\perp \mid U) \subseteq \mathscr{A}^\infty(\mu_\perp)$. The reverse inclusion being trivial, we are done. ▮

The proof of this theorem is an adaptation of a proof from Conway and Olin's $P^\infty(\mu)$ paper [CO, p. 28]. The author has simply replaced the interior of the Sarason hull for $P^\infty(\mu)$ in their paper by U, the domain of analyticity of Chaumat's map applied to f.

Clearly $\mathbb{C}\backslash \bar{E} \subseteq U$, so the part of $\mathscr{A}^\infty(\mu_\perp)$ living over $\mathbb{C}\backslash \bar{E}$ is all L^∞. This is saying nothing however by III.2(c)! From III.2(c) and (d) and III.4, we immediately get the following.

III.5. COROLLARY. *If $\check{f}$ is nonconstant on each component of E^0, then* $\mathscr{A}^\infty(\mu_\perp) = \mathscr{A}^\infty(\mu_\perp \mid \partial E) \oplus L^\infty(\mu_\perp \mid E^0)$.

Thus, assuming $\check{f}$ nonconstant on each component of E^0, we now seek to determine when $\mathscr{A}^\infty(\mu_\perp \mid \partial E) = L^\infty(\mu_\perp \mid \partial E)$. For many K and μ, this occurs because $R^\infty(K, \mu_\perp \mid \partial E) = L^\infty(\mu_\perp \mid \partial E)$. Two examples where this phenomenon occurs and one example where it does not follow. In all three examples, $K = \{\frac{1}{2} \leqslant |z| \leqslant 1\}$, $\mu_\perp = \mu$, and $E = K^0$.

(a) $\mu =$ area measure on K: $R^\infty(K, \mu_\perp \mid \partial E) = L^\infty(\mu_\perp \mid \partial E)$ trivially since $\mu_\perp \mid \partial E = 0$.

(b) $\mu =$ area measure on $\{\frac{1}{2} \leqslant |z| \leqslant \frac{3}{4}\}$ + arclength measure on $\{|z| = 1\}$: $R^\infty(K, \mu_\perp \mid \partial E) = L^\infty(\mu_\perp \mid \partial E)$ by the Stone–Weierstrass theorem since $\mu_\perp | \partial E =$ arclength measure on $\{|z| = 1\}$ and $1/z = \bar{z}$ on $\{|z| = 1\}$.

(c) $\mu =$ arclength measure on ∂K: $R^\infty(K, \mu_\perp \mid \partial E) \neq L^\infty(\mu_\perp \mid \partial E)$ since $\mu_\perp \mid \partial E = \mu_\perp$ and $R^\infty(K, \mu_\perp)$ has no nontrivial L^∞ summand.

With a little more work some functions in the R^∞ space of example (c) can be dealt with. Let X be a compact Hausdorff space and let A be a uniform algebra on X. A point $x_0 \in X$ is called a *peak point* for A iff there exists a $g \in A$ such that $g(x_0) = 1$ and $|g(x)| < 1$ for each $x \in X\backslash\{x_0\}$. Call a set $C \subseteq \mathbb{C}$ *almost open* iff the area of $C\backslash C^0$ is zero.

III.6. COROLLARY. *Assume the set of nonpeak points for $R(K)$ is almost open. Suppose that each component of E^0 has a boundary point with a neighborhood onto which $\check{f}$ has a nonconstant analytic extension. Then* $\mathscr{A}^\infty(\mu_\perp) = L^\infty(\mu_\perp)$.

Proof. Let ν be a weak* continuous annihilator of $\mathscr{A}^\infty(\mu_\perp)$. Recall that the Cauchy transform of ν is defined by the equation $\hat{\nu}(z) \equiv \int d\nu(\zeta)/(\zeta - z)$ for all z such that the defining integral converges absolutely (which is for area–a.e. $z \in \mathbb{C}$ [G, page 46]). Since $\nu \perp R(K)$, the measure $\mu_z(\zeta) \equiv \nu(\zeta)/\hat{\nu}(z)(\zeta - z)$ places $z \in E$ whenever $\hat{\nu}(z) \neq 0$ [G, proof of II.8.5]. Thus $\hat{\nu} = 0$ area–a.e. off E. In particular, $\hat{\nu} = 0$ everywhere off $\bar{E}$ (III.2(c)).

Let C be a component of E^0 and let Δ be an open disc about a point $z_0 \in \partial C$ such that f has a nonconstant analytic extension to Δ. Then $C \cup \Delta$ is an open connected set contained in U. By III.4, $|\nu|(U)=0$. Thus $\hat{\nu}$ is clearly analytic on U and so too on $C \cup \Delta$. A simple topological argument utilizing III.2(b) shows $(C \cup \Delta)\backslash \bar{E}$ to be nonempty. Thus $\hat{\nu}$ vanishes on a nonempty open subset of $C \cup \Delta$. It follows that $\hat{\nu}$ vanishes on all of $C \cup \Delta$ and so all of C.

From the last two paragraphs we conclude that $\hat{\nu}=0$ area–a.e. off $E\backslash E^0$. Since the set of nonpeak points for $R(K)$ is assumed almost open, E is also almost open [D, II.13]. Hence $\hat{\nu}=0$-a.e. on $\mathbb{C}$ and so $\nu=0$ [G, II.8.3]. By the Hahn–Banach theorem, we are done. ∎

The maximum modulus principle implies that points in K^0 are nonpeak points for $R(K)$. Hence the set of nonpeak points for $R(K)$ is almost open whenever ∂K has zero area. Thus this last corollary disposes of the K and μ in example (c) whenever $\check{f}$ has a nonconstant analytic extension to a neighborhood of some point of ∂K. This leaves open those cases where $\check{f}$ does not so extend. As the $R(K)$ in example (c) is hypodirichlet (by IV.4 below), those cases will be dealt with in the next section. In preparation for this, this section closes with a number of items leading to a refinement of III.5 and an item that will yield us L^∞ summands.

Recall that two complex homomorphisms ϕ and ψ of a uniform algebra A are said to *belong to the same part* of A iff there is a finite positive number C such that $1/C < (\operatorname{Re}\phi(g)/\operatorname{Re}\psi(g)) < C$ for each $g \in A$ with $\operatorname{Re} g > 0$. Belonging to the same part of A is evidently an equivalence relation on the maximal ideal space of A. The resulting equivalence classes are called *parts* of A. A part is said to be *trivial* iff it is a singleton. The part of any peak point is trivial (as can be seen by contemplating $g_n \equiv ((1+g)/2)^n$ where $g \in A$ is any function peaking at the point).

For any bounded Borel set $C \subseteq \mathbb{C}$, define $R(C)$ to be the uniform closure in $C(\bar{C})$ of the functions $g(z) \equiv \int (g^*(\zeta)/(\zeta - z))\, d\,\text{area}\,(\zeta)$, where g^* ranges over all compactly supported, bounded, Borel functions on $\mathbb{C}$ that vanish area–a.e. on C. Fubini's theorem shows that a measure ν on $\bar{C}$ annihilates $R(C)$ iff $\hat{\nu}=0$ area–a.e. off C. Consequently, if C is compact, this definition of $R(C)$ agrees with our old one. Our interest is in $R(C)$ for $C=E$ only.

III.7. PROPOSITION. (a) *$R(E)$ is a uniform algebra on $\bar{E}$ containing $R(\bar{E})$ all of whose functions are analytic on E^0* [Ch, III.1].

(b) *The maximal ideal space of $R(E)$ is $\bar{E}$* [Ch, VI. 3].

(c) *The set of complex homomorphisms of $R(E)$ whose parts are nontrivial = the set of nonpeak points for $R(E)=E$.* [Ch, VI.6 *and its proof*].

Let A be a uniform algebra on a compact Hausdorff space X. Recall that a measure ν on X is said to *represent* a complex homomorphism ϕ of A iff $\phi(g)=\int g\,d\nu$ for each $g\in A$. Such a ν is called a *complex representing measure* for ϕ on A. If ν is also positive, it is called a *representing measure* for ϕ on A. We apologize for the fact that a measure which represents need not be a representing measure!

III.8. LEMMA. *Suppose $\nu\perp R(K)$, $\nu\ll\mu$, and $\hat{\nu}(z)\neq 0$. Then $z\in E$ and there exists a representing measure m for z on $R(E)$ such that $m\ll\nu$ and $m(\{z\})=0$.*

Proof. By [G, proof of II.8.5], $(1/\hat{\nu}(z))(\nu(\zeta)/(\zeta-z))$ is a complex representing measure for z on $R(K)$. By [G, II.2.2], there is a representing measure m for z on $R(K)$ that is absolutely continuous with respect to $(1/\hat{\nu}(z))(\nu(\zeta)/(\zeta-z))$. Clearly $m\ll\nu$ and $m(\{z\})=0$. Since $\nu\ll\mu$, m shows that $z\in E$. By [Ch, III.3], m is a representing measure for z on $R(E)$. ∎

III.9. THEOREM. *The set of nontrivial parts of $R(E)$ is countable. Let $\{E_n\}$ be an ennumeration of these parts. Then $\mu_\perp=\sum_n\mu_n$, where the μ_n's are pairwise singular nonzero measures annihilating $R(E)$ with the property that for each $z\in E_n$, there exists a representing measure m for z on $R(E)$ such that $\mu_n\ll m$. Furthermore, $R^\infty(K,\mu_\perp)=\sum_n R^\infty(K,\mu_n)$ and $\mathscr{A}^\infty(\mu_\perp)=\sum_n\mathscr{A}^\infty(\mu_n)$.*

Proof. By III.8, $\widehat{\mu_\perp}=0$ area–a.e. off E and so $\mu_\perp\in R(E)^\perp$. Let $\{E_\alpha\}$ be the collection of all parts of $R(E)$, nontrivial or otherwise. Select a point z_α from each E_α. By [G, VI.2.3], $\mu_\perp=\mu_s+\sum_\alpha\mu_\alpha$, where μ_s and the μ_α's are pairwise singular measures annihilating $R(E)$, μ_s is singular to all representing measures for $R(E)$, and each μ_α is absolutely continuous with respect to some representing measure m_α for z_α on $R(E)$.

Since μ_s is singular to all representing measures for $R(E)$, III.8 implies that $\hat{\mu}_s=0$ area–a.e. on $\mathbb{C}$ and so $\mu_s=0$ [G, II.8.3].

Let $z\in E_\alpha$. By [G, VI.1.2], there is a representing measure m for z on $R(E)$ such that $m_\alpha\ll m$. Thus $\mu_\alpha\ll m$.

Evidently $\mu_\alpha\neq 0$ for at most countably many α. Thus to establish all but the last sentence of the theorem it suffices to show that $\mu_\alpha\neq 0$ iff the part of z_α is nontrivial.

Suppose the part of z_α is trivial. By III.7(c), z_α is a peak point for $R(E)$. By [G, II.11.3], $m_\alpha=\delta_{z_\alpha}$, the point mass at z_α. Hence $\mu_\alpha\ll\delta_{z_\alpha}$. As $\mu_{\alpha'}$ annihilates the constants, it must be that $\mu_\alpha=0$.

Suppose the part of z_α is nontrivial. By III.7(c), $z_\alpha\in E$. Letting μ_{z_α} be as in the definition of E and setting $\nu(\zeta)\equiv(\zeta-z_\alpha)\,\mu_{z_\alpha}(\zeta)$, we have $\hat{\nu}(z_\alpha)\neq 0$. By III.8, there is a representing measure m for z_α on $R(E)$ such that $m\ll\nu$. Thus $m\ll\mu=\sum_{\alpha'}\mu_{\alpha'}$. For $\alpha'\neq\alpha$, $m_{\alpha'}$ is singular to m [G, VI.2.2]. Since $\mu_\alpha\ll m_{\alpha'}$, it follows that $m\ll\mu_\alpha$ and so $\mu_\alpha\neq 0$.

Discard those α such that $\mu_\alpha = 0$ and change each remaining α to an n. Select pairwise disjoint sets Δ_n such that $\mu_n = \mu_\perp | \Delta_n$. Let ν be a weak* continuous annihilator of $R^\infty(K, \mu_\perp)$. By III.8, $\hat{\nu} = 0$ area–a.e. off E and so $\nu \perp R(E)$. Set $\nu_n \equiv \nu | \Delta_n$. Clearly $\nu_n \ll m_n$. Since the $m_n s$ are representing measures for points in different parts of $R(E)$, from [G, VI.2.2] it follows that $\nu - \nu_n = \sum_{m \neq n} \nu_m$ is singular to all representing measures for z_n on $R(E)$. Thus $\nu = \nu_n + (\nu - \nu_n)$ is the Lebesgue decomposition of ν with respect to the set of all representing measures for z_n on $R(E)$ [G, II.7.4; II.7.5]. By the abstract F. and M. Riesz theorem [G, II.7.6], $\nu_n \perp R(E)$. Clearly $\nu_n \perp R(K)$ and $\nu_n \ll \mu_n$, i.e., ν_n is a weak* continuous annihilator of $R^\infty(K, \mu_n)$. But then $\nu = \sum_n \nu_n$ is a weak* continuous annihilator of $\sum_n R^\infty(K, \mu_n)$. By the Hahn–Banach theorem, $\sum_n R^\infty(K, \mu_n) \subseteq R^\infty(K, \mu_\perp)$. As the reverse inclusion is trivial, equality holds.

Notice that $\chi_{\Delta_n} \in \sum_n R^\infty(K, \mu_n) = R^\infty(K, \mu_\perp) \subseteq \mathscr{A}^\infty(\mu_\perp)$. As $\mathscr{A}^\infty(\mu_\perp)$ is a weak* closed subalgebra of $L^\infty(\mu_\perp)$, $\mathscr{A}^\infty(\mu_n) = \mathscr{A}^\infty(\mu_\perp | \Delta_n) = \overline{\chi_{\Delta_n} \mathscr{A}^\infty(\mu_\perp)}^{wk^* L^\infty(\mu_\perp)} \subseteq \mathscr{A}^\infty(\mu_\perp)$. Hence $\sum_n \mathscr{A}^\infty(\mu_n) \subseteq \mathscr{A}^\infty(\mu_\perp)$. As the reverse inclusion is trivial, equality holds. ∎

For now and forevermore, let E_n and μ_n be as in III.9. From III.5 and III.9, we immediately get the following refinement of III.5.

III.10. Corollary. *If $\check{f}$ is nonconstant on each component of E^0, then* $\mathscr{A}^\infty(\mu_\perp) = \sum_n \mathscr{A}^\infty(\mu_n | \partial E) \oplus L^\infty(\mu_n | E^0)$.

Thus, assuming $\check{f}$ nonconstant on each component of E^0, we now seek to determine when $\mathscr{A}^\infty(\mu_n | \partial E) = L^\infty(\mu_n | \partial E)$. The last item of this section in conjunction with the rather meager supply of representing measures for a hypodirichlet algebra (see IV.1 below) shall be our means of forcing $\mathscr{A}^\infty(\mu_n | \partial E) = L^\infty(\mu_n | \partial E)$ in the next section.

III.11. Lemma. *Suppose $\nu \ll \mu_n | \partial E$. Then either $R^\infty(K, \nu) = L^\infty(\nu)$ or there exists a point $z \in E_n$ and two representing measures m and m' for z on $R(E)$, both supported on ∂E, such that $m \ll \nu \ll m'$.*

Proof. Suppose $R^\infty(K, \nu) \neq L^\infty(\nu)$. Then by the Hahn–Banach theorem, there is a nonzero measure $\sigma \ll \nu$ such that $\sigma \perp R(K)$. For some z, $\hat{\sigma}(z) \neq 0$ [G, II.8.3]. By III.8, $z \in E$ and there exists a representing measure m for z on $R(E)$ such that $m \ll \sigma$. Thus $m \ll \nu \ll \mu_n$. By III.9, there exist representing measures for points in E_n not singular to m. Hence $z \in E_n$ [G, VI.2.2]. By III.9 again, there is a representing measure $\tilde{m}$ for z on $R(E)$ such that $\nu \ll \mu_n \ll \tilde{m}$.

By III.7(a), the functions in $R(E)$ are continuous on $\bar{E}$ and analytic on E^0. Thus the maximum modulus principle implies that the functional $\Lambda(g | \partial E) \equiv \int_{E^0} g \, d\tilde{m}$ is well-defined from $R(E) | \partial E$ to $\mathbb{C}$. Clearly $\Lambda(1) =$

$\tilde{m}(E^0)=\|A\|$. Let τ be a measure on ∂E arising from a norm-preserving linear extension of A to all of $C(\partial E)$. Then $\int d\tau=\tilde{m}(E^0)=\int d|\tau|$, so τ is positive. Set $m'\equiv\tilde{m}\,|\,\partial E+\tau$. Clearly m', supported on ∂E, is a representing measure for z on $R(E)$ such that $\nu\ll m'$. ∎

IV. The Hypodirichlet Case

Let A be a uniform algebra on a compact Hausdorff space X. A *logmodular measure* for a complex homomorphism ϕ of A is a positive measure m on X such that $\log|\phi(g)|=\int\log|g|\;dm$ for each $g\in A^{-1}$. Each logmodular measure for ϕ is also a representing measure for ϕ (given $g\in A$, consider e^g and $e^{ig}\in A^{-1}$). Just as representing measures for ϕ are in one-to-one correspondence with the norm-preserving linear extensions of the functional $\operatorname{Re} g\to\operatorname{Re}\phi(g)$ from $\operatorname{Re} A$ to C_RX, so too logmodular measures for ϕ are in one-to-one correspondence with the norm-preserving linear extensions of the functional $\sum_n\alpha_n\log|g_n|\to\sum_n\alpha_n\log|\phi(g_n)|$ from the real linear span of $\log|A^{-1}|$ to $C_R(X)$. We call A *hypodirichlet* iff the uniform closure of $\operatorname{Re} A$ in $C_R(X)$ has finite codimension and the real linear span of $\log|A^{-1}|$ is uniformly dense in $C_R(X)$. Thus if A is hypodirichlet, ϕ has a finite-dimensional set of representing measures and a unique logmodular measure.

The single result we need from the general theory of hypodirichlet algebras follows. As the author can find this result in only one place in the literature and there it is stated as an exercise [G, exercise 11 from Chap. IV], a proof is provided.

IV.1. Theorem. *If A is hypodirichlet and ϕ is a complex homomorphism of A, then all representing measures for ϕ are mutually absolutely continuous.*

Proof. Let m_ϕ denote the unique logmodular measure for ϕ and let m be any representing measure for ϕ. It must be shown that $m\approx m_\phi$. That $m\ll m_\phi$ follows easily from the literature, see [S, 23.13 and the proof of 26.32 with obvious minor changes] or [G, IV.4.1; IV.5.2; IV.7.5]. The reader will probably find the items of the first reference more self-contained than those of the second.

Denote the weak* closure in $L^\infty(m)$ of A by $H^\infty(m)$. Set $\mathscr{M}\equiv$ the maximal ideal space of $L^\infty(m)$ and look upon $H^\infty(m)$ as residing in $C(\mathscr{M})$. Extend ϕ to a complex homomorphism on $H^\infty(m)$ be setting $\phi(g)\equiv\int g\,dm$ for each $g\in H^\infty(m)$. Define a linear functional L on the real linear span of $\log|H^\infty(m)^{-1}|$ by $L(\sum_n\alpha_n\log|g_n|)\equiv\sum_n\alpha_n\log|\phi(g_n)|$. The argument of [G, IV.7.1] shows that L is well-defined with norm one. Taking a norm-preserving linear extension of L to all of $C_R(\mathscr{M})$, we get a measure ν on $\mathscr{M}$

with $\|\nu\|=1$ such that $\log|\phi(g)|=\int \log|g|\,d\nu$ for each $g\in H^{\infty-1}(m)$. Since $\int d\nu=1=\int d|\nu|$, ν is a positive measure. For $g\in H^\infty(m)$, $\operatorname{Re}\int g\,d\nu=\int \operatorname{Re} g\,d\nu=\int \log|e^g|\,d\nu=\log|\phi(e^g)|=\log|e^{\phi(g)}|=\operatorname{Re}\phi(g)$. Similarly $\operatorname{Im}\int g\,d\nu=\operatorname{Im}\phi(g)$, so $\phi(g)=\int g\,d\nu$. Although $H^\infty(m)$ need not be a uniform algebra on $\mathscr{M}$ (it may fail to separate points), retaining the uniform algebra terminology we call ν a representing measure on $\mathscr{M}$ for the complex homomorphism ϕ of $H^\infty(m)$. Consider that $L^1(m)\subseteq L^1(m)^{**}=L^\infty(m)^*=C(\mathscr{M})^*=$ the space of measures on $\mathscr{M}$. Now [G, IV.2.3] asserts that the set of representing measures on $\mathscr{M}$ for the complex homomorphism ϕ of $H^\infty(m)$ is the weak* closure in the space of measures on $\mathscr{M}$ of the set of representing measures in $L^1(m)$ for the complex homomorphism ϕ of A. As the latter set of representing measures is finite-dimensional by the hypodirichlicity of A, it is also weak* closed. Thus ν is actually a positive measure on X such that $\nu\ll m$. Clearly $\log|\phi(g)|=\int \log|g|\,d\nu$ for each $g\in A^{-1}$, so ν is a logmodular measure for ϕ on A. As there is only one such measure, namely m_ϕ, we conclude that $m_\phi\ll m$. ∎

Leaving the general theory, note that $R(K)$ is not hypodirichlet whenever $K^0\neq\varnothing$ (consider any nonzero function from $C_R(K)$ that vanishes on ∂K and use the maximum modulus principle to show that it is not uniformly approximable on K by real linear combinations of functions from $\log|R(K)^{-1}|$). However, by the maximum modulus principle, $R(K)$ is isometricly isomorphic to $R(K)|\partial K$ under the obvious restriction map and so, abusing language vigorously, we shall say that $R(K)$ is *hypodirichlet* when we really mean that $R(K)|\partial K$ is hypodirichlet.

Gamelin and Garnett have found several necessary and sufficient conditions for $R(K)$ to be hypodirichlet [GG1]. To state these a few definitions are necessary. We say $R(K)$ is *strongly pointwise boundedly dense* in $H^\infty(K^0)$ iff for each bounded analytic function g on K^0, there exists a sequence $\{g_n\}\subseteq R(K)$ with $\|g_n\|_K\leqslant\|g\|_{K^0}$ such that $g_n\to g$ pointwise on K^0. The *analytic capacity* of a subset C of $\mathbb{C}$, denoted $\gamma(C)$, is the supremum of $|g'(\infty)|$, where g ranges over all functions analytic and bounded in modulus by one off some compact subset K_g of C.

IV.2. THEOREM. *The following are equivalent*:

(a) *$R(K)$ is hypodirichlet*;

(b) *$\mathbb{C}\backslash K^0$ has finitely many components, $R(\partial K)=C(\partial K)$, and $R(K)$ is strongly pointwise boundedly dense in $H^\infty(K^0)$*;

(c) *$\mathbb{C}\backslash K^0$ has finitely many components and for each $z\in\partial K$ and each $0<\delta<$ diameter of the component of $\mathbb{C}\backslash K^0$ containing z, $\gamma(\Delta(z;\delta)\backslash K)\geqslant\delta/4$; and*

(d) $\mathbb{C}\backslash K^0$ *has finitely many components and for each* $z \in \partial K$,

$$\liminf_{\delta \downarrow 0} \frac{\gamma(\Delta(z;\delta)\backslash K)}{\delta} > 0.$$

Furthermore, when finite, the codimension of $\operatorname{Re} R(K)|\partial K$ *in* $C_R(\partial K)$ *is equal to the number of bounded components of* $\mathbb{C}\backslash K^0$.

Two comments are necessary concerning IV.2. First, [GG1] is merely an announcement of results with no proofs. Proofs can be found in [GG2] but then only for the case where $R(K)$ is *dirichlet*, i.e., only for the case where $\operatorname{Re} R(K)|\partial K$ is uniformly dense in $C_R(K)$. How to handle the hypodirichlet case is sketched in Section 11 of [GG2], however. Second, IV.2 as stated does not appear in [GG1]. Theorem 7 of [GG1] is close but not the same. Accordingly we now indicate how to get IV.2 from [GG1] and some other sources. To see that (a) implies (b) use theorems 1 and 7 from [GG1], [G, VIII.11.1], and IV.1 above. To see that (b) implies (c) use Theorem 4 from [GG1], some point-set topology, and [G, VIII.2.1]. That (c) implies (d) is trivial. The rest of IV.2 follows from Theorem 7 of [GG1].

Our first use of IV.2 will be to identify the nonpeak points and nontrivial parts of a hypodirichlet $R(K)$. The maximal ideal space of $R(K)$ is easily seen to be just K itself and so the parts of $R(K)$ form a partition of K.

IV.3. Corollary. *If* $R(K)$ *is hypodirichlet, then the set of complex homomorphisms of* $R(K)$ *whose parts are nontrivial = the set of nonpeak points for* $R(K) = K^0$ *and the nontrivial parts of* $R(K)$ *are precisely the components of* K^0.

Proof. As noted previously, for any uniform algebra, the part of a peak point is trivial. By [G, VI.3.1], for any $R(K)$, the part of a nonpeak point is nontrivial. By Harnack's inequality and a connectedness argument, for any $R(K)$, each component of K^0 is contained in a nontrivial part. Hence it needs only be shown that for any hypodirichlet $R(K)$, ∂K consists of peak points for $R(K)$ and different components of K^0 lie in different parts of $R(K)$.

Let $z \in \partial K$. By IV.1, δ_z is the only representing measure for z on $R(K)|\partial K$. Hence by [G, II.11.3], z is a peak point for $R(K)|\partial K$. By the maximum modulus principle, z is a peak point for $R(K)$.

Let C be a component of K^0. Since χ_C is a bounded analytic function on K^0, by IV.2(b) there exists a sequence $\{g_n\} \subseteq R(K)$ with $\|g_n\|_K \leqslant 1$ such that $g_n \to \chi_C$ pointwise on K^0. By [G, VI.2.1] points from C and $K^0\backslash C$ belong to different parts of $R(K)$. ▮

Our second use of IV.2 will be to prove an inheritance result, namely that $R(E)=R(\bar{E})$ is hypodirichlet whenever $R(K)$ is hypodirichlet. A definition and three corollaries are necessary first however. These corollaries are also useful for constructing hypodirichlet $R(K)$. Call $R(K)$ *n-hypodirichlet* iff $R(K)$ is hypodirichlet and $\mathbb{C}\backslash K^0$ has $\leqslant n$ bounded components. By IV.2, $R(K)$ is n-hypodirichlet iff $R(K)$ is hypodirichlet with the codimension of $\operatorname{Re} R(K)|\partial K$ in $C_R(\partial K)$ being $\leqslant n$. In particular, $R(K)$ is 0-hypodirichlet iff it is dirichlet.

IV.4. COROLLARY. *If $\mathbb{C}\backslash K$ has n bounded components, then $R(K)$ is n-hypodirichlet.*

Proof. Clearly $\mathbb{C}\backslash K^0=\overline{\mathbb{C}\backslash K}$ has $\leqslant n$ bounded components. Point-set topology shows that for each $z\in\partial K$ and each $\delta>0$ sufficiently small, $\Delta(z;\delta)\backslash K$ contains a continuum of diameter $\geqslant\delta/2$. By [G, VIII.2.1], $\gamma(\Delta(z;\delta)\backslash K)\geqslant\delta/8$. Hence

$$\liminf_{\delta\downarrow 0}\frac{\gamma(\Delta(z;\delta)\backslash K)}{\delta}>0.$$

Having verified IV.2(d) for K, we are done. ∎

IV.5. COROLLARY. *If $R(K)$ is n-hypodirichlet and J is a compact subset of K such that the closure of each component of $K\backslash J$ meets ∂K, then $R(J)$ is n-hypodirichlet.*

Proof. The hypotheses on K and J and some point-set topology show that $\mathbb{C}\backslash J^0$ has $\leqslant n$ bounded components. Suppose $z\in\partial J$. If $z\in\partial K$, then by IV.2(d) applied to K,

$$\liminf_{\delta\downarrow 0}\frac{\gamma(\Delta(z;\delta)\backslash J)}{\delta}\geqslant\liminf_{\delta\downarrow 0}\frac{\gamma(\Delta(z;\delta)\backslash K)}{\delta}>0.$$

If $z\in K^0$, the hypothesis on J and some point-set topology show that for each $0<\delta<$ distance of z to ∂K, $\Delta(z;\delta)\backslash J$ contains a continuum of diameter $\geqslant\delta/2$. By [G, VIII.2.1], $\gamma(\Delta(z;\delta)\backslash J)\geqslant\delta/8$. Hence

$$\liminf_{\delta\downarrow 0}\frac{\gamma(\Delta(z;\delta)\backslash J)}{\delta}>0.$$

Having verified IV.2(d) for J, we are done. ∎

IV.6. COROLLARY. *If $\{K_m\}$ is a decreasing sequence of compact sets such that each $R(K_m)$ is n-hypodirichlet and $K\equiv\bigcap_m K_m$, then $R(K)$ is n-hypodirichlet.*

Proof. The hypotheses on the K_m's and some point-set topology show that $\mathbb{C}\backslash K^0$ has $\leqslant n$ bounded components and that for each $z \in \partial K$, there exist $M \geqslant 1$, $\delta_0 > 0$, and a sequence $\{z_m\}_{m \geqslant M}$ converging to z with each $z_m \in \partial K_m$ such that the diameter of the component of $\mathbb{C}\backslash K_m^0$ containing z_m is $\geqslant \delta_0$. Given $0 < \delta < \delta_0$, choose $m \geqslant M$ such that $z_m \in \Delta(z; \delta/2)$. By IV.3(c) applied to K_m, $\gamma(\Delta(z, \delta)\backslash K) \geqslant \gamma(\Delta(z_m; \delta/2)\backslash K) \geqslant \delta/8$. Hence

$$\liminf_{\delta \downarrow 0} \frac{\gamma(\Delta(z; \delta)\backslash K)}{\delta} > 0.$$

Having verified IV.2(d) for K, we are done. ∎

Corollaries IV.5 and IV.6 in the case where $n = 0$ can be found in [GG2] as 9.6 and 9.7. Indeed our proofs of IV.5 and IV.6 are obvious modifications of the proofs in [GG2] of 9.6 and 9.7.

Taking the text of [D] from just after the statement of II.11 up to the end of the proof of II.12 and replacing each occurance of "Dirichlet," "II.2(d)," "II.11(b)," "II.11(c)," and "μ" by "n-hypodirichlet," "III.2(c)," "IV.5," "IV.6," and "$\mu_\perp$," respectively, a proof of the following result is obtained.

IV.7. THEOREM. *$R(\bar{E} \cap L)$ is n-hypodirichlet whenever L is a closed set such that $R(K \cap L)$ is n-hypodirichlet.*

Clearly IV.7 contains the inheritance result needed. Combining this with previous results, a rather complete and nice description of $R^\infty(K, \mu_\perp)$ is obtained for the case where $R(K)$ is hypodirichlet. For an open subset U of $\mathbb{C}$, let $H^\infty(U)$ denote the space of all bounded analytic functions on U equipped with the supremum norm.

IV.8. COROLLARY. *Suppose $R(K)$ is hypodirichlet. Then $R(E) = R(\bar{E})$ is hypodirichlet, $E = (\bar{E})^0$ is open, $\{E_n\}$ is precisely the collection of components of E, and Chaumat's map $\check{}$ is an isometric isomorphism of $R^\infty(K, \mu_\perp) = \sum_n R^\infty(K, \mu_n)$ onto $H^\infty(E) = \sum_n H^\infty(E_n)$ which carries each $R^\infty(K, \mu_n)$ onto $H^\infty(E_n)$.*

Proof. By IV.7, $R(\bar{E})$ is hypodirichlet. Let $\nu \perp R(\bar{E})$ and suppose $\hat{\nu}(z) \neq 0$. By [G, proof of II.8.5], $(1/\hat{\nu}(z))(\nu(\zeta)/(\zeta - z))$ is a complex representing measure for z on $R(\bar{E})$ putting no mass on z. By [G, II.11.3], z is a nonpeak point for $R(\bar{E})$. By III.2(b) and IV.3, $z \in (\bar{E})^0 = E^0 \subseteq E$. Thus $\hat{\nu} = 0$ area–a.e. off E, i.e., $\nu \perp R(E)$. The Hahn–Banach theorem and III.7(a) now imply that $R(E) = R(\bar{E})$. The assertions concerning E and $\{E_n\}$ now follow from III.7(c) and IV.3.

Clearly $\check{}$ is always a contractive homomorphism. By III.2(d), $\check{}$ is into

$H^\infty(E)$. Given $g \in H^\infty(E)$, because of IV.2(b) applied to $R(\bar{E})$, we may choose $\{g_n\} \subseteq R(E)$ with $\|g_n\|_{\bar{E}} \leqslant \|g\|_E$ such that $g_n \to g$ pointwise on E. By III.8, if ν is a weak* continuous annihilator of $R^\infty(K, \mu_\perp)$, then $\hat{\nu} = 0$ area–a.e. off E, i.e., $\nu \perp R(E)$. By the Hahn–Banach theorem, $R(E) \subseteq R^\infty(K, \mu_\perp)$ and so via III.2(a) we see that $\{g_n\} \subseteq \|g\|_E$ ball $R^\infty(K, \mu_\perp)$. Let $h \in \|g\|_E$ ball $R^\infty(K, \mu_\perp)$ be a weak* cluster point of $\{g_n\}$. Then for each $z \in E$, $\check{h}(z)$ is a cluster point of $\{\check{g}_n(z)\}$. The map $k \to \check{k}(z)$ is a complex homomorphism on $R^\infty(K, \mu_\perp)$ and so too on $R(E) \subseteq R^\infty(K, \mu_\perp)$. Hence via III.7(b) we see that $\check{g}_n(z) = g_n(z)$. As $g_n(z) \to g(z)$, $\check{h}(z) = g(z)$. Note that $\|h\|_{\mu_\perp} \leqslant \|g\|_E = \|\check{h}\|_E$. Thus $\check{}$ is an isometry onto $H^\infty(E)$.

Let $z \in E_n$. Setting $\nu(\zeta) \equiv (\zeta - z)\ \mu_z(\zeta)$, applying III.8, and taking the m that results and calling it μ_z, we may aassume that μ_z is a representing measure for z on $R(E)$. But then by [G, VI.2.2] and III.9, μ_z is singular to each $\mu_{n'}$ with $n' \neq n$. Since $\mu_z \ll \mu_\perp = \sum_{n'} \mu_{n'}$, it follows that $\mu_z \ll \mu_n$. Thus given $z \in E_n$, μ_z can always be chosen absolutely continuous with respect to μ_n. From this it easily follows that $\check{}$ carries $R^\infty(K, \mu_n)$ onto $H^\infty(E_n)$. ∎

We are now in a position to solve the weak* density problem in the case where $R(K)$ is hypodirichlet.

IV.9. THEOREM. *If $R(K)$ is hypodirichlet, then the following are equivalent*:

(a) $\mathscr{A}^\infty(\mu_\perp) = L^\infty(\mu_\perp)$,

(b) *f is nonconstant in each space $L^\infty(\mu_n)$, and*

(c) *$\check{f}$ is nonconstant on each set E_n.*

Proof. (a) ⇒ (b): Suppose $f \equiv c$ in some space $L^\infty(\mu_n)$. Then $\mathscr{A}^\infty(\mu_n) = R^\infty(K, \mu_n)$. Since μ_n itself is a weak* continuous annihilator of $R^\infty(K, \mu_n)$, $R^\infty(K, \mu_n) \neq L^\infty(\mu_n)$. Hence $\mathscr{A}^\infty(\mu_\perp) = \sum_{n'} \mathscr{A}^\infty(\mu_{n'}) \neq \sum_{n'} L^\infty(\mu_{n'}) = L^\infty(\mu_\perp)$.

(b) ⇒ (c): We prove a slightly stronger fact which will be needed below when proving (b) ⇒ (a), namely that f is constant in $L^\infty(\mu_n)$ whenever $\check{f}$ is constant area-a.e. on E_n. So suppose $\check{f} \equiv c$ area-a.e. on E_n. Since $\mu_n \perp R(K)$ and $f \in R^\infty(K, \mu_n)$, $f\mu_n \perp R(K)$. By III.8, if $(f\mu_n)\hat{}(z) \neq 0$, then $z \in E$ and there exists a representing measure m for z on $R(E)$ such that $m \ll f\mu_n$. But then by III.9 and [G, VI.2.2], $z \in E_n$. Hence $(f\mu_n)\hat{} = 0$ area-a.e. off E_n. Similarly $(c\mu_n)\hat{} = 0$ area-a.e. off E_n. We conclude that $(f\mu_n)\hat{} = (c\mu_n)\hat{}$ area-a.e. off E_n.

Let $z \in E_n$ be such that $\int (d|\mu_n|(\zeta))/(|\zeta - z|) < \infty$. Choose a net $\{f_\alpha\}$ of rational functions with poles off K converging to f weak* in $L^\infty(\mu_n)$. Then

$$\int \frac{f_\alpha(\zeta) - f_\alpha(z)}{\zeta - z}\, d\mu_n(\zeta) \to \int \frac{f(\zeta) - \check{f}(z)}{\zeta - z}\, d\mu_n(\zeta).$$

But each $(f_\alpha(\zeta)-f_\alpha(z))/(\zeta-z)$ being a rational function with poles off K integrates against μ_n to give zero. Hence $\int ((f(\zeta)-\check{f}(z))/(\zeta-z))\, d\mu_n(\zeta)=0$, i.e., $(f\mu_n)\hat{}(z)=\check{f}(z)\ \hat{\mu}_n(z)$. We conclude that $(f\mu_n)\hat{}=\check{f}\hat{\mu}_n=c\hat{\mu}_n=(c\mu_n)\hat{}$ area-a.e. on E_n.

The last two paragraphs imply that $(f\mu_n)\hat{}=(c\mu_n)\hat{}$ area-a.e. on $\mathbb{C}$. By [G, 8.3], $f\mu_n=c\mu_n$, i.e., $f\equiv c$ in $L^\infty(\mu_n)$.

(c) $\Rightarrow$ (b): Suppose $f\equiv c$ in $L^\infty(\mu_n)$. To conclude that $\check{f}\equiv c$ on E_n it suffices to know that given $z\in E_n$, one may choose $\mu_z\ll\mu_n$. But this was shown in the last paragraph of the proof of IV.8 (and without using hypodirichlicity).

(b) $\Rightarrow$ (a): Since we have shown (b) $\Rightarrow$ (c), by III.10 and IV.8 it suffices to show that $\mathscr{A}^\infty(\mu_n|\partial E)=L^\infty(\mu_n|\partial E)$. Clearly this occurs whenever $R^\infty(K,\mu_n|\partial E)=L^\infty(\mu_n|\partial E)$, so we may as well assume that $R^\infty(K,\mu_n|\partial E)\neq L^\infty(\mu_n|\partial E)$. But then by III.11, some $z\in E_n$ has two representing measures m and m' on $R(E)$, both supported on ∂E, such that $m\ll\mu_n|\partial E\ll m'$. By IV.1 and IV.8, $\mu_n|\partial E\approx m$. Thus we must show that $\mathscr{A}^\infty(m)=L^\infty(m)$.

Suppose that $f\equiv c$ in $L^\infty(m)$. Given $w\in E_n$, let $\tilde{m}$ be a representing measure for w on $R(E)$ supported on ∂E. By [G, VI.1.2], IV.1, and IV.8, $\tilde{m}\ll m$. Thus $f\equiv c$ in $L^\infty(\tilde{m})$. If $m(\{w\})=0$, we may take $\mu_w=\tilde{m}$ and conclude that $\check{f}(w)=\int f d\mu_w=\int c d\tilde{m}=c$. Since $m(\{w\})=0$ for at most countably many w, $\check{f}\equiv c$ area-a.e. on E_n. By the proof above of (b) $\Rightarrow$ (c), $f\equiv c$ in $L^\infty(\mu_n)$, a contradiction. Thus f is nonconstant in $L^\infty(m)$.

Let $g\equiv \operatorname{Re} f$ or $\operatorname{Im} f$ and be nonconstant in $L^\infty(m)$. Choose $\alpha\in R$ such that for $\Delta\equiv\{g\leqslant\alpha\}$, $m(\Delta)>0$ and $m(\mathbb{C}\backslash\Delta)>0$. Select a sequence $\{u_n\}\subseteq C_R([-\|g\|,+\|g\|])$ with $\|u_n\|=1$ such that $u_n\equiv 1$ on $[-\|g\|,\alpha]$ and $u_n\downarrow 0$ pointwise on $(\alpha,+\|g\|]$. Since f and $\bar{f}\in\mathscr{A}^\infty(m)$, $g\in\mathscr{A}^\infty(m)$. Since each u_n can be approximated uniformly on $[-\|g\|,+\|g\|]$ by polynomials in z, each $u_n\circ g\in\mathscr{A}^\infty(m)$. Since $u_n\circ g\to\chi_\Delta$ pointwise boundedly and so weak* in $L^\infty(m)$, $\chi_\Delta\in\mathscr{A}^\infty(m)$. But $\mathscr{A}^\infty_{(m)}$ is a weak* closed subalgebra of $L^\infty(m)$, so $\mathscr{A}^\infty(m|\Delta)=\overline{\chi_\Delta\mathscr{A}^\infty(m)}^{wk^*L^\infty(m)}\subseteq\mathscr{A}^\infty(m)$. Similarly $\mathscr{A}^\infty(m|\mathbb{C}\backslash\Delta)\subseteq\mathscr{A}^\infty(m)$. Hence $\mathscr{A}^\infty(m)=\mathscr{A}^\infty(m|\Delta)\oplus\mathscr{A}^\infty(m|\mathbb{C}\backslash\Delta)$.

Suppose $R^\infty(K,m|\Delta)\neq L^\infty(m|\Delta)$. Then by III.11, there exists a point $w\in E_n$ and a representing measure $\tilde{m}$ for w on $R(E)$ supported on ∂E such that $\tilde{m}\ll m|\Delta$. By [G, VI.1.2], IV.1, and IV.8, $m\ll\tilde{m}$. Thus $m\ll m|\Delta$. This contradicts $m(\mathbb{C}\backslash\Delta)>0$. Hence $R^\infty(K,m|\Delta)=L^\infty(m|\Delta)$ and so $\mathscr{A}^\infty(m|\Delta)=L^\infty(m|\Delta)$. Similarly $\mathscr{A}^\infty(m|\mathbb{C}\backslash\Delta)=L^\infty(m|\mathbb{C}\backslash\Delta)$ and so $\mathscr{A}^\infty(m)=L^\infty(m)$. ∎

An inspection of the proof of IV.9 reveals that hypodirichlicity was only used in proving (b) $\Rightarrow$ (a), so (a) $\Rightarrow$ (b) and (b) $\Leftrightarrow$ (c) hold for arbitrary K.

If one is not concerned with minimizing the use of hypodirichlicity in proving IV.9, then (b) $\Leftrightarrow$ (c) can be gotten immediately from IV.8.

If one is not concerned with having a proof with possibilities for adaptation beyond the hypodirichlet case and if one does not mind using a result coming at the end of a long and difficult paper that depends essentially on many earlier results of that paper, then the proof of (b) $\Rightarrow$ (a) of IV.9 can be shortened. The paper is [AS] and the result [Lemma 14.5 (and Section 15)] asserts that $R^\infty(K, m)$ is a maximal proper weak* closed subalgebra of $L^\infty(m)$ for $R(K)$ hypodirichlet and m a representing measure supported on ∂K for a complex homomorphism of $R(K)$.

In proving (b) $\Rightarrow$ (a) of IV.9, the hypodirichlicity of $R(K)$ forced the hypodirichlicity of $R(E)$ which then was used two ways only, first, to force the E_n's to be open and connected, and, second, to force the representing measures supported on ∂E for a complex homomorphism of $R(E)$ to be mutually absolutely continuous.

V. Open Questions and Relevant Examples

Is the conclusion of IV.9 true for more general K? It is convenient to discuss this first assuming the set of nonpeak points for $R(K)$ to be almost open. In this case, E itself is almost open [D, II.13] and we break our original question in two.

First Question. Suppose E is almost open and $\check{f}$ is nonconstant on each nontrivial part of $R(E)$. Is $\check{f}$ nonconstant on each component of E^0?

Second Question. Suppose E is almost open and $\check{f}$ is nonconstant on each component of E^0. Is $\mathscr{A}^\infty(\mu_\perp) = L^\infty(\mu_\perp)$?

If $R(E)$ is hypodirichlet, the first question has an affirmative answer for trivial reasons. The simplest example where the first question has an affirmative answer for nontrivial reasons arises when K is a string of beads [G, p. 146, Fig. 3] and μ is area measure on K or arclength measure on the outer boundary of K. Here E consists of one nontrivial part of $R(E)$ and, modulo a subset of $K \cap R$, is simply K^0, which has two components C_+ and C_-. The affirmative answer follows easily once one uses H^p theory to realize that $\check{f} | C_+$ and $\check{f} | C_-$ have boundary values a.e. on $K \cap R$ and that these boundary values on $K \cap R$ must agree a.e. Incidentally, from this fact we readily see that $R(K)$ is not strongly pointwise boundedly dense in $H^\infty(K^0)$ and so by IV.2(b), $R(E) = R(K)$ is not hypodirichlet.

Below we present a theorem yielding an affirmative answer to the second question for many K and μ with $R(K)$ not hypodirichlet. Then we construct a K and μ for which the answer to the second question appears to be unknown at present.

Say that a component C of K^0 *kisses* a circle $\partial\Delta$ iff either $K \subseteq \bar{\Delta}$ and C

contains some $S \equiv \{z+\rho^{i\theta}: r'<\rho<r,\ \alpha<\theta<\beta\}$ or $K \subseteq \mathbb{C}\backslash\Delta$ and C contains some $S \equiv \{z+\rho^{i\theta}: r<\rho<r',\ \alpha<\theta<\beta\}$, where $\Delta=\Delta(z;r)$.

V.1. THEOREM. *Suppose the set of nonpeak points for $R(K)$ is almost open and K^0 has only finitely many components all of which kiss circles. Let μ be such that $\bar{E}=K$ and let $\check{f}$ be nonconstant on each component of E^0. Then $\mathscr{A}^\infty(\mu_\perp)=L^\infty(\mu_\perp)$.*

Proof. Note that $\bar{E}=K$ and III.2(b) imply that $E^0=K^0$ and $\partial E=\partial K$. Hence by III.5 it suffices to show that $\mathscr{A}^\infty(\mu_\perp \mid \partial K)=L^\infty(\mu_\perp \mid \partial K)$.

Let the components of K^0 be denoted $C_1,\ldots, C_n$. Each C_i kisses a circle $\partial\Delta_i$ along an annular sector S_i as in the definition above. Set $I_i \equiv$ the interior in $\partial\Delta_i$ of $\bar{S}_i \cap \partial\Delta_i$. Abusing language somewhat, for any subset C of a rectifiable curve, let $l(C)$ denote the arclength of C and let $l \mid C$ denote arclength measure on C.

Since $\check{f} \mid S_1$ is a nonconstant bounded analytic function, by H^p theory $\check{f} \mid S_1$ has boundary values F defined $l \mid \partial S_1$-a.e. and F is nonconstant in $L^\infty(l \mid I_1)$. A hard theorem of Chaumat [Ch, IV.3] allows us to choose $\{f_n\} \subseteq R(E)$ uniformly bounded in $L^\infty(\mu_\perp)$ converging to f $\mu_\perp$-a.e. Then $\{\check{f}_n\}$ converges pointwise boundedly to $\check{f}$. An argument in the second paragraph of the proof of IV.8 applies here to show that $\check{f}_n=f_n \mid E$. Hence by III.7(a), $\{f_n \mid \bar{S}_1\}$ is a sequence of functions continuous on $\bar{S}_1$ and analytic on S_1 converging pointwise boundedly on S_1 to $\check{f} \mid S_1$. By H^p theory, $f_n \to F$ weak* in $L^\infty(l \mid \partial S_1)$. We conclude that f is nonconstant in $L^\infty(\mu_\perp \mid I_1)$ whenever $\mu_\perp \mid I_1 \approx l \mid \partial S_1$.

Thus if $\mu_\perp \mid I_1 \approx l \mid I_1$, by arguing as in the third paragraph of the proof of (b) $\Rightarrow$ (a) of IV.9, we get a set $\Delta \subseteq \mathbb{C}$ such that for $v_1 \equiv \mu_\perp \mid \partial K \cap \Delta$ and $v_2 \equiv \mu_\perp \mid \partial K\backslash\Delta$, $\mathscr{A}^\infty(\mu_\perp \mid \partial K)=\sum_{j=1}^{2} \mathscr{A}^\infty(v_j)$ with $v_j \mid I_1 \not\approx l \mid I_1$ for $j=1,2$. If $\mu_\perp \mid I_1 \not\approx l \mid I_1$, the same conclusion can be gotten by setting $v_1 \equiv \mu_\perp \mid \partial K$ and $v_2 \equiv 0$. Hence in any case, the conclusion follows.

Split v_1 and v_2 over I_2 just as we split $\mu_\perp \mid \partial K$ over I_1. Then take the four pieces of $\mu_\perp \mid \partial K$ that result and split them over I_3 just as we split $\mu_\perp \mid \partial K$ over I_1. Continue up to I_n. Relabelling the resulting pieces of $\mu_\perp \mid \partial K$, we see that $\mathscr{A}^\infty(\mu_\perp \mid \partial K)=\sum_{j=1}^{2^n} \mathscr{A}^\infty(v_j)$ with $v_j \mid I_i \not\approx l \mid I_i$ for $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant 2^n$. It thus suffices to show that each $\mathscr{A}^\infty(v_j)=L^\infty(v_j)$. We do this by showing each $R^\infty(K, v_j)=L^\infty(v_j)$.

Fix i and j. Suppose J is a compact subset of $\partial\Delta_i$ such that $l(J)=0$. By [G, II.12.6], there is a function g continuous on $\bar{\Delta}_i$ and analytic on $\bar{\Delta}_i$ such that $g \equiv 1$ on J and $|g|<1$ on $\bar{\Delta}_i\backslash J$. Assume, as we may, that $K \subseteq \bar{\Delta}_i$. Then $g \in R(K)$ since any function continuous on $\bar{\Delta}_i$ and analytic on Δ_i can be approximated uniformly on $\bar{\Delta}_i$ by polynomials in z. As $\mu_\perp \in R(K)^\perp$, $\int g^n\, d\mu_\perp=0$ for each n. But $g^n \to \chi_J$ pointwise boundedly on $\bar{\Delta}_i$, so $\mu_\perp(J)=0$. We conclude that $\mu_\perp \mid \partial\Delta_i \ll l \mid \partial\Delta_i$.

But then $v_j|I_i \ll l|I_i$ and so it must be that $l|I_i \not\ll v_j|I_i$. Choose L a compact subset of I_i such that $|v_j|(L)=0$ and $l(L)>0$. Let u be the harmonic function on Δ_i with boundary values χ_L on $\partial\Delta_i$ and let u^* be the harmonic conjugate of u on Δ_i. Set $h \equiv e^{u+iu^*}$. Then $h \in H^\infty(\Delta_i)$ and its boundary values H on $\partial\Delta_i$ satisfy

$$|H| = \begin{cases} e & l\text{-a.e. on } L \\ 1 & l\text{-a.e. on } \partial\Delta_i \backslash L. \end{cases}$$

Let $\{p_n\}$ be a sequence of polynomials in z uniformly bounded in $L^\infty(l|\partial\Delta_i)$ and converging weak* in $L^\infty(l|\partial\Delta_i)$ to H (the Cesaro means of H will do nicely). Set

$$k \equiv \begin{cases} h & \text{on } \Delta_i \\ H & l\text{-a.e. on } \partial\Delta_i. \end{cases}$$

As $p_n \to h$ pointwise boundedly on Δ_i and $v_j|\partial\Delta_i \ll l|\partial\Delta_i$, $p_n \to k$ weak* in $L^\infty(v_j)$ thus forcing $k \in R^\infty(K, v_j)$.

Let $\mathscr{E}$ be the envelope of v_j with respect to K. For each $z \in \mathscr{E} \cap \Delta_i$, $h(z) = \lim_{n\to\infty} p_n(z) = \lim_{n\to\infty} \check{p}_n(z) = \check{k}(z)$. Hence $\|h\|_{\mathscr{E}\cap\Delta_i} = \|\check{k}\|_{\mathscr{E}\cap\Delta_i} \leqslant \|k\|_{\mathscr{E}} \leqslant \|k\|_{v_j} = \max\{\|h\|_{\Delta_i\backslash S_i}, \|H\|_{l|\partial\Delta_i\backslash L}\} < \|H\|_{S_i}$. Clearly $S_i \nsubseteq \mathscr{E}$ and so $C_i \nsubseteq \mathscr{E}$. By III.2(c), $\mathscr{E} \cap C_i = \varnothing$. Now letting i vary, we conclude that $\mathscr{E} \cap K^0 = \varnothing$. Denote the set of nonpeak points for $R(K)$ by Q. The definition of the envelope and [G, II.11.3] imply that $\mathscr{E} \subseteq Q$. Clearly $Q^0 = K^0$. Thus $\mathscr{E} \subseteq Q\backslash Q^0$. We've assumed $Q\backslash Q^0$ has area zero, so $\mathscr{E}$, too, has area zero.

Let σ be a weak* continuous annihilator of $R^\infty(K, v_j)$. By III.8, $\hat{\sigma} = 0$ area-a.e. off $\mathscr{E}$ and so area-a.e. on $\mathbb{C}$. By [G, II.8.3], $\sigma = 0$. By the Hahn–Banach theorem, we are done. ▮

Theorem V.1 answers our second question affirmatively for a number of K and μ with $R(E) = R(K)$ not hypodirichlet, e.g., K a roadrunner set [G, p. 52, Fig. 2], a champagne bubble set [G, p. 227, Fig. 6], or a string of beads, and μ arclength measure on the outer boundary of K.

Theorem V.1 can be strengthened considerably. Given Γ a rectifiable Jordan curve and C a connected open subset of $\mathbb{C}\backslash\Gamma$, call a point $p \in \partial C \cap \Gamma$ a *strong boundary point* of C with respect to Γ iff Γ has a normal at p and there exists an open isosceles triangle $T \subseteq C$ with p being the vertex of T common to the two sides of T of equal length and with the normal of Γ through p bisecting the interior angle of T at p. Say that a component C of K^0 *kisses* Γ iff K is contained in the closure of one of the two components of $\mathbb{C}\backslash\Gamma$ and the set of strong boundary points of C with respect to Γ has positive arclength as a subset of Γ. Then the proof of V.1, some conformal mapping theory as found in [K, II.C and D], and a construction from page 130 of [OT] can be combined to yield a proof of the following theorem. As the resulting proof is tedious and technical, it is omitted.

V.2. THEOREM. *Suppose the set of nonpeak points for $R(K)$ is almost open and K^0 has only finitely many components all of which kiss rectifiable Jordan curves. Let μ be such that $\bar{E}=K$ and let $\tilde{f}$ be nonconstant on each component of E^0. Then $\mathscr{A}^\infty(\mu_\perp)=L^\infty(\mu_\perp)$.*

One comment seems called for. The definition of a strong boundary point given here differs from that given in [OT]. With a little work however, it can be shown that the set of strong boundary points defined one way has positive arclength measure iff the set of strong boundary points defined the other way does. That is all that really matters here.

We now construct a K and μ not falling within the scope of any of our results. Given an annulus $A=\Delta(p;R)\backslash\overline{\Delta(p;r)}$ and a number $0<\alpha<1$, call $\{\Delta(z_n;r_n)\}$ a *suitable inner (outer) α-collection of discs* for A iff

(a) $\{z_n\}$ is a sequence of distinct points in A with $|z_n-p|\to r(R)$ such that every point of $\partial\Delta(p;r)(\partial\Delta(p;R))$ can be approached by points from $\{z_n\}$ nontangentially through sectors of arbitrarily small angular opening, and

(b) $\{r_n\}$ is a sequence of positive numbers such that the discs $\overline{\Delta(z_n;2r_n)}$ are all pairwise disjoint, contained in A, with radii summing to $<\alpha$.

Set $K_0\equiv\overline{\Delta(0;1)}$. Let $\{\Delta(z_n^{(1)};r_n^{(1)})\}$ be a suitable outer α-collection of discs for $\Delta(0;1)\backslash\overline{\Delta(0;1/2)}$ with $\alpha=\frac{1}{2}$. Set $K_1\equiv K_0\backslash\bigcup_n\Delta(z_n^{(1)};r_n^{(1)})$. For each n, let $\{\Delta(z_m^{[n]};r_m^{[n]})\}_m$ be a suitable inner α_n-collection of discs for $\Delta(z_n^{(1)};2r_n^{(1)})\backslash\Delta(z_n^{(1)};r_n^{(1)})$, where $\sum_n\alpha_n<\frac{1}{4}$. Relabel $\{\Delta(z_m^{[n]};r_m^{[n]})\}_{n,m}$ as $\{\Delta(z_n^{(2)};r_n^{(2)})\}$ and set $K_2\equiv K_1\backslash\bigcup_n\Delta(z_n^{(2)};r_n^{(2)})$. Continue in this fashion to get a decreasing sequence of compact sets $\{K_m\}$. Set $K\equiv\bigcap_m K_m$ and $\mu\equiv$ arclength measure on the outer boundary of K. One may appropriately call K a very fizzy champagne bubble set.

Note that $K^0\subseteq E\subseteq K$ and ∂K has area zero, so E is almost open and our second question applies. Since $\mu=\mu_\perp=\mu_\perp|\partial E$, III.5 is of no use here. Since $R(K)$ is not hypodirichlet, IV.9 is of no use here. Although the set of nonpeak points for $R(K)$ is almost open, K^0 has only one component, and $\bar{E}=K$, V.2 is of no use here since K^0 kisses no rectifiable Jordan curves.

Third Question. For the K and μ just constructed, is $\mathscr{A}^\infty(\mu_\perp)=L^\infty(\mu_\perp)$ whenever $\tilde{f}$ is nonconstant on each component of E^0?

Turning to the remaining case where the set of nonpeak points for $R(K)$ is not almost open, we exhibit a K showing that the conclusion of IV.9 can fail. The K in question is gotten from a Swiss cheese J due to McKissick. For any Swiss cheese, $R(J) \neq C(J)$ yet $J^0 = \varnothing$ [G, p. 26, Fig. 1]. McKissick's amazing Swiss cheese in addition has the property that $R(J)$ is *normal*, i.e., given disjoint compact subsets J_1 and J_2 of J, there exists a function $g \in R(J)$ such that $g \equiv 0$ on J_1 and $g \equiv 1$ on J_2 [S, 27.4].

Since $R(J) \neq C(J)$, a result of Bishop [G, II.11.4] implies that the set of nonpeak points for $R(J)$ is nonempty. By [G, VI.3.1], $R(J)$ has a nontrivial part Q with positive area. Set $K \equiv \bar{Q}$ and $\mu \equiv$ area measure on Q.

Clearly $K^0 = \varnothing$. Let $z \in Q$. By [G, II.11.3], we may choose a representing measure $m \neq \delta_z$ for z on $R(J)$. By [G, VI.3.3], m is supported on K and represents z on $R(K)$. But then $\nu(\zeta) \equiv (\zeta - z)\, m(\zeta)$ is a nonzero annihilating measure for $R(K)$, so $R(K) \neq C(K)$. Clearly $R(K)$ is normal. By [G, VI.1.1, VI.2.2, and VI.3.3], Q is contained in a single part of $R(K)$. A hard result due to Melnikov, Theorem 1 from [M], states that $\bar{Q} \backslash Q$ consists of peak points for $R(J)$ and thus $R(K)$. It follows that $R(K)$ has exactly one nontrivial part, namely Q.

By [G, II.11.3], $E \subseteq Q$. A hard result due to Gamelin and Garnett, Theorem 1.3 from [GG3], now implies that $E = Q$. Since $K^0 = \varnothing$, $E^0 = \varnothing$ and E is thus far from being almost open. Let $\nu \perp R(K)$. If $\hat{\nu}(z) \neq 0$, then by [G, proof of II.8.5] $(1/\hat{\nu}(z))(\nu(\zeta)/(\zeta - z))$ represents z on $R(K)$, and so by [G, II.11.3] $z \in Q$. Thus $\hat{\nu} = 0$ area-a.e. off $Q = E$, i.e., $\nu \perp R(E)$. By the Hahn–Banach theorem and III.7(a), $R(E) = R(K)$. Hence $R(E)$ has exactly one nontrivial part, namely E.

Fix $p \in E$. Since E has positive area, we may choose $0 < r < R$ such that for $\Delta \equiv \Delta(p; r)$ and $\Delta' \equiv \Delta(p; R)$, the areas of both $E \cap \Delta$ and $E \backslash \overline{\Delta'}$ are both positive. Since $K^0 = \varnothing$, we may select $\{z_n\} \subseteq \mathbb{C} \backslash K$ converging to p. Set $f_n(\zeta) \equiv 1/(\zeta - z_n)$. Then

$$|f_n(p)| \leqslant \|f_n\|_{K \backslash \Delta} \|\mu_p | K \backslash \Delta\| + \|f_n\|_{K \cap \Delta} \|\mu_p | K \cap \Delta\|.$$

Since $|f_n(p)| \to \infty$ and $\|f_n\|_{K \backslash \Delta} \|\mu_p | K \backslash \Delta\|$ remains bounded as $n \to \infty$, it must be that $\|\mu_p | K \cap \Delta\| \neq 0$. Thus any μ_p puts mass on Δ.

Fix a μ_p and set $\nu(\zeta) \equiv (\zeta - p)\, \mu_p(\zeta)$. By a result of Bishop [G, II.10.2], ν can be written as $\nu_1 + \nu_2$, where $\nu_1 \in R(K)^\perp$ has support in $K \cap \Delta'$ and $\nu_2 \in R(K)^\perp$ has support in $K \backslash \bar{\Delta}$. Clearly ν_1 must put mass on Δ and so is nonzero. By [G, II.10.1 and the proof of II.10.2], $\nu_1 = h\nu - (1/\pi)\, \bar{\partial} h \hat{\nu}$ (area), where h is a compactly supported continuously differentiable function. But $\hat{\nu} = 0$ area-a.e. off $E = Q$ by III.8. Hence $\nu_1 \ll \mu$. By III.1, $\nu_1 \ll \mu_\perp$. Thus we have constructed a nonzero weak* continuous annihilator ν_1 of $R^\infty(K, \mu_\perp)$ with support in $K \cap \Delta'$.

Let $f \in R(K)$ be such that $f \equiv 0$ on spt $\nu_1 \cup (K \cap \bar{\Delta})$ and $f \equiv 1$ on $K \backslash \Delta'$. Clearly f satisfies (b) and (c) of IV.9. However, $\mathscr{A}^\infty(\mu_\perp) | \operatorname{spt} \nu_1 =$

$R^{\infty}(K, \mu_{\perp})|\operatorname{spt} \nu_1$. It follows that ν_1 is a nonzero weak* continuous annihilator of $\mathscr{A}^{\infty}(\mu_{\perp})$ and so $\mathscr{A}^{\infty}(\mu_{\perp}) \neq L^{\infty}(\mu_{\perp})$. Thus (a) of IV.9 fails.

In this example, although $\check{f}$ is nonconstant on the single nontrivial part of $R(E)$, it is constant on a subset of positive area in that part. This suggests the following.

Fourth Question. For K and μ arbitrary, is $\mathscr{A}^{\infty}(\mu_{\perp}) = L^{\infty}(\mu_{\perp})$ iff each level set of $\check{f}$ has area zero?

Of course, if E is almost open and $\check{f}$ is nonconstant on each component of E^0, then by analyticity each level set of $\check{f}$ will be a countable subset of E^0 union a subset of $E \backslash E^0$ and so have area zero. Conversely, if each level set of $\check{f}$ has area zero, then $\check{f}$ is nonconstant on each component of E^0. Thus when E is almost open, the "if" part of the fourth question becomes the second question.

Incidently, abbreviating the condition on $\check{f}$ in the fourth question by (c′), the last paragraph and IV.8 show that (c) from IV.9 $\Leftrightarrow$ (c′) when $R(K)$ is hypodirichlet. We also know that (b) from IV.9 $\Leftrightarrow$ (c) from IV.9 always. The obvious questions arise. Abbreviate the condition that each level set of f have $\mu_{\perp}$-measure zero by (b′). Is (b) from IV.9 $\Leftrightarrow$ (b′) when $R(K)$ is hypodirichlet? Is (b′) $\Leftrightarrow$ (c′) always? The answer is "no" on both counts since one can have $R(K)$ dirichlet and $\mu_{\perp} = \mu$ totally atomic. As an example of this phenomenon consider $K = \overline{\Delta(0; 1)}$ and $\mu = \sum_n \delta_{z_n}/2^n$ where $\{z_n\}$ is a sequence of distinct points in $\Delta(0; 1)$ with $|z_n| \to 1$ such that every point of $\partial\Delta(0; 1)$ can be approached nontangentially by points from $\{z_n\}$. Thus for K and μ arbitrary, there appears to be no natural condition on f, as opposed to $\check{f}$, with a chance of being equivalent to $\mathscr{A}^{\infty}(\mu_{\perp}) = L^{\infty}(\mu_{\perp})$.

With regard to the "only if" part of the fourth question, the obvious way to show that $\mathscr{A}^{\infty}(\mu_{\perp}) \neq L^{\infty}(\mu_{\perp})$ given a level set of $\check{f}$ of positive area is to construct a nonzero weak* continuous annihilating measure of $R^{\infty}(K, \mu_{\perp})$ concentrated on that level set. Assuming that area measure on E is absolutely continuous with respect to μ and that a level set of $\check{f}$ has interior relative to E, this can be done by modifying the argument, hinging on [G, II.10.2], just given for the example above.

Finally, we have had occasion to use the fact that $R(E)$ inherits certain properties of $R(K)$. This enabled us to state certain hypotheses directly in terms of $R(K)$ (as in III.6 and IV.9). When properties do not pass from $R(K)$ to $R(E)$ or when we are ignorant as to whether they do or not, then we must also put hypotheses on K and μ that force $R(E) = R(K)$ (as in V.1 and V.2) or simply state hypotheses in terms of $R(E)$. This inconvenience, and natural curiosity, suggest the following.

Fifth Question. Which properties does $R(E)$ inherit from $R(K)$ and which does it not?

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Erratum. The following lines were inadvertently omitted from [D]. They should begin the first paragraph on page 379: "Consequently we may suppose that $z_n \in \sigma_e(S)$ for only finitely many n. Dropping these n, we may assume that $\{z_n\}_{n \geqslant 1} \subseteq E \backslash \sigma_e(S) \subseteq \sigma(S) \backslash \sigma_e(S)$."